

HÖLDER REGULARITY OF THE SOLUTION TO THE COMPLEX MONGE-AMPÈRE EQUATION WITH L^p DENSITY

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ABSTRACT. On a smooth domain $\Omega \subset \subset \mathbb{C}^n$, we consider the Dirichlet problem for the complex Monge-Ampère equation $((dd^c u)^n = f dV, u|_{b\Omega} \equiv \phi)$. We state the Hölder regularity of the solution u when the boundary value ϕ is Hölder and the density f is only L^p . Note that, in former literature (Kolodziej and Guedj-Kolodziej-Zeriahi) the weakness of the assumption $f \in L^p$ was balanced by taking $\phi \in C^{1,1}$ (in addition to assuming Ω strongly pseudoconvex).

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1. INTRODUCTION

For a bounded pseudoconvex domain $\Omega \subset \subset \mathbb{C}^n$, the Dirichlet problem for the Monge-Ampère equation consists in

$$(1.1) \quad \begin{cases} (dd^c u)^n = f dV & \text{on } \Omega, \\ u|_{b\Omega} = \varphi|_{b\Omega}. \end{cases}$$

In our discussion, we take a density $0 \leq f \in L^p(\Omega)$, $1 \leq p \leq +\infty$, a boundary datum $\varphi \in C^\alpha(b\Omega)$, and look for a plurisubharmonic solution $u \in C^\beta(\bar{\Omega})$ for a certain β . Sometimes, we use the notation $MA(\varphi, f)$ for the problem (1.1) and $u(\varphi, f)$ for its solution. This problem has been extensively investigated in recent years under the assumption that Ω is strongly pseudoconvex. [3], [20] and [2] show that there is a solution $u \in C^0(\bar{\Omega})$ if $\varphi \in C^0(b\Omega)$, $f \in C^0(\bar{\Omega})$. By the celebrated “comparison principle” (cf. [13]), the solution is unique; what matters is to prove the Hölder continuity of this C^0 -solution. In this direction, [2] proves that $u \in C^{\frac{\alpha}{2}}(\bar{\Omega})$ if $\varphi \in C^\alpha(b\Omega)$, $f^{\frac{1}{n}} \in C^{\frac{\alpha}{2}}(\bar{\Omega})$. A recent interest has been dedicated to the case when Ω is no longer strongly pseudoconvex but has a certain “finite type” $2m$. [17] proves that $u \in C^{\frac{\alpha}{2m}}(\bar{\Omega})$ if $\varphi \in C^\alpha$, $f^{\frac{1}{n}} \in C^{\frac{\alpha}{2m}}$. [8] gets the same conclusion with a more geometric notion of finite type $2m$ (cf. (1.2) below) and has also a generalization

for the infinite type. Coming back to the case of Ω strongly pseudoconvex, [5] proves that $u \in C^\infty$, for $\varphi, f \in C^\infty$, in case $f > 0$. Lowering the smoothness of f , gives the problem additional difficulty. [7] proves that if $f \in L^p$ and $\varphi \in C^{1,1}$, then $u \in C^{2\gamma}$ for $\gamma = \gamma_p := \frac{1}{qn+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$. Our purpose is twofold: to lower the regularity of φ and to allow a (geometric) finite type $2m$ for Ω . What we get is that if $f \in L^p$, $\varphi \in C^\alpha$, then $u \in C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}$. To go into the detail of our geometric setting, we consider a submanifold $S \subset b\Omega$ of CR dimension 0, denote by d_S the distance to S and by $(L_{b\Omega})$ the Levi form of $b\Omega$. We assume that $b\Omega$ has finite type $2m$ along S in the sense that

$$(1.2) \quad L_{b\Omega} \gtrsim d_S^{2m-2}.$$

To convert (1.2) into a suitable property for our use, we need two basic results. First, from [11], we know that (1.2) implies the potential-theoretic “ $t^{\frac{1}{m}}$ -property”. By [10] and [8] this implies in turn that there is an exhaustion function ρ which defines Ω by $\rho < 0$ such that

$$(1.3) \quad \partial\bar{\partial}\rho \geq \text{Id} \text{ on } \Omega, \quad \rho \in C^{\frac{1}{m}}.$$

Remark 1.1. According to Catlin [6], if Ω has finite D’Angelo type D , then it has the “ $t^{\frac{1}{m}}$ -property” for $\frac{1}{m} := D^{-n^2 D^{n^2}}$; again, this implies the existence of the exhaustion $\rho \in C^{D^{-n^2 D^{n^2}}}$ with $\partial\bar{\partial}\rho \geq \text{Id}$.

It is (1.3) the property which rules many passages of the paper. Here is our result

Theorem 1.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth domain of finite type $2m$ in the sense of (1.2) and let $\varphi \in C^\alpha(b\Omega)$ with $0 < m\alpha \leq 2$ and $f \in L^p(\Omega)$. Then for the unique plurisubharmonic continuous solution u to $MA(\varphi, f)$ we have*

$$(1.4) \quad u \in C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}, \text{ for } \gamma = \gamma_p := \frac{1}{qn+1} \text{ where } \frac{1}{q} + \frac{1}{p} = 1.$$

The proof follows in Section 3. Note that, in particular, if Ω is strongly pseudoconvex and $\alpha \leq 2\gamma$, then $u \in C^{\alpha\gamma}$.

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2. HÖLDER REGULARITY OF A SUBSOLUTION TO MA

We recall a classical result for $\varphi \in C^0$ and $f \in L^p$.

Proposition 2.1. *Let ρ satisfy (1.3). Then there is a subsolution $v \in C^0$ for $\varphi \in C^0$ and $f \in L^p$.*

Proof. We define

$$\tilde{f} := \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } \mathbb{B}^n \setminus \Omega. \end{cases}$$

We consider the solutions

$$\begin{cases} u_1 = u(\mathbb{B}^n, 0, \tilde{f}) \in C^0 & \text{by Kolodjei on the ball } \mathbb{B}^n \text{ (strongly pseudoconvex) [14],} \\ u_2 = u(\Omega, (-u_1)|_{b\Omega}, 0) \in C^0 & \text{by Blocki [1].} \end{cases}$$

Taking summation $u_1 + u_2$ we have a subsolution to $MA(\Omega, 0, f) \in C^0$. Using the solution $u(\Omega, \varphi, 0) \in C^0$ provided by [1], and putting

$$v = u_1 + u_2 + u(\Omega, \varphi, 0),$$

we get the desired subsolution. \square

We change a little our setting and take $\varphi \in C^\alpha$ and $f \in L^\infty$. If ζ is a general point of $b\Omega$, we set

$$(2.1) \quad v_\zeta = \begin{cases} \varphi(\zeta) - c[-\rho(z) + |z - \zeta|^2]^{\alpha m} & \text{if } 0 < \alpha m \leq 1, \\ \varphi(\zeta) - \sum_j 2\operatorname{Re} \frac{\partial \varphi}{\partial z_j}(\zeta)(z_j - \zeta_j) - c[-\rho(z) + |z - \zeta|^2]^{\alpha m} & \text{if } 1 < \alpha m \leq 2. \end{cases}$$

If there is an exhaustion function $\rho \in C^{\frac{1}{m}}$ such that $\partial\bar{\partial}\rho \geq \operatorname{Id}$, then we can find c , independent of ζ and only depending on $\|\varphi\|_\alpha$ and $\|f^{\frac{1}{n}}\|$ such that (cf. [17])

$$(2.2) \quad \begin{cases} v_\zeta(z) \leq \varphi(z) \text{ for any } z \in b\Omega, & v_\zeta(\zeta) = \varphi(\zeta), \\ (dd^c v_\zeta)^n \geq f, \\ v_\zeta \in C^{\frac{\alpha}{m}}. \end{cases}$$

Using the family $\{v_\zeta\}_{\zeta \in b\Omega}$, it is readily seen (cf. [17]) that for any plurisubharmonic C^0 solution to MA, we have $u(\varphi, f) \in C^{\frac{\alpha}{m}}$ for $\varphi \in C^\alpha$ and $f^{\frac{1}{n}} \in C^{\frac{\alpha}{m}}$; in particular, $u(\varphi, 0) \in C^{\frac{\alpha}{m}}$ for $\varphi \in C^\alpha$. We lower the smoothness of f . We start from

Proposition 2.2. *Let ρ satisfy (1.3) and let $0 < m\alpha \leq 2$. Then there is a subsolution $v \in C^{\frac{\alpha}{m}}$ for $\varphi \in C^\alpha$ and $f \in L^\infty$.*

Proof. We consider the solution $u(\varphi, 0) \in C^{\frac{\alpha}{m}}$ by Li and Ha-Khanh and define

$$v = u(\varphi, 0) + c\rho.$$

For $c \geq \|f\|_{L^\infty}$, v is a subsolution. \square

We now take $f \in L^p$.

Proposition 2.3. *Let ρ satisfy (1.3) and let $0 < m\alpha \leq 2$. Then there is a subsolution $v \in C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}$ for $\varphi \in C^\alpha$ and $f \in L^p$.*

Proof. We define \tilde{f} by f on Ω and 0 on $\mathbb{B}^n \setminus \Omega$, and consider the solutions

$$\begin{cases} u_1 = u(\mathbb{B}^n, 0, \tilde{f}) \in C^{2\gamma} & \text{by Guedi, Kolodjei and Zeriahi [7],} \\ u_2 = u(\Omega, (-u_1)|_{\partial\Omega}, 0) \in C^{\frac{2\gamma}{m}} & \text{by Li [17] and Ha-Khanh [8].} \end{cases}$$

Note here that [7] can be applied since the boundary datum, which is 0, has the required $C^{1,1}$ regularity. Taking a solution $u(\Omega, \varphi, 0) \in C^{\frac{\alpha}{m}}$ (cf. [17] and taking summation $v = u_1 + u_2 + u(\Omega, \varphi, 0)$ we have the desired subsolution $v \in C^{\min(\frac{\alpha}{m}, \frac{2\gamma}{m})}$. □

3. HÖLDER REGULARITY OF A SOLUTION - PROOF OF THEOREM 1.2

We recall a crucial fact from [14]. For a general domain, not necessarily strongly pseudoconvex, the existence of $u(\varphi, 0) \in C^0$ (which turns out to be equivalent to the existence of a maximal function with boundary datum φ), in addition to the existence of a subsolution $v \in C^0$ for $\varphi \in C^0$ and $f \in L^p$, implies the existence of a solution $u(\varphi, f) \in L^\infty$ for $f \in L^p$. In particular,

Theorem 3.1. *(Kolodziej [14]) Assume Ω is defined by $\rho < 0$ for $\rho \in C^0(\bar{\Omega})$ such that $\partial\bar{\partial}\rho \geq \text{Id}$. Then for any $\varphi \in C^0$, $f \in L^p$ there is a (unique) plurisubharmonic solution $u(\varphi, f) \in L^\infty$.*

Proof. By the property of ρ , which implies b-regularity, there is a solution for continuous data, in particular for $f = 0$, that is $u(\varphi, 0)$ (cf. [1]); thus there is a maximal function for given boundary data. Again by the property of ρ , there is a subsolution for $\varphi \in C^0$, $f \in L^p$ (Proposition 2.1 above). Then by [14] Thm. C p. 97 (3 lines after the statement) there is a solution in L^∞ . □

Remark 3.2. The solution $u(\varphi, f)$, $\varphi \in C^0$, $f \in L^p$ is in fact in C^0 by [12]. Note that the paper makes the general assumption of pseudoconvexity of Ω but this is needless for this specific conclusion. This is confirmed by private communication of the author.

We assume from now on $\partial\bar{\partial}\rho \geq \text{Id}$ for $\rho \in C^{\frac{1}{m}}$.

According to Proposition 2.3 above, when we take a smoother boundary datum $\varphi \in C^\alpha$, there is a subsolution $v \in C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}$ for $f \in L^p$.

What follows is dedicated to show that, in this situation, the L^∞ plurisubharmonic solution $u(\varphi, f)$ is in fact in $C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}$.

First it is immediate to find a supersolution w , analogous to the subsolution v ; comparison principle yields at once

$$(3.1) \quad v \leq u(\varphi, f) \leq w.$$

By (3.1) and by the $C^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}$ regularity of v and w we get

$$|u(z) - u(\zeta)| \lesssim |z - \zeta|^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}, \quad z \in \bar{\Omega}, \quad \zeta \in b\Omega,$$

and therefore for δ suitably small

$$(3.2) \quad |u(z) - u(z')| \lesssim \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}, \quad z, z' \in \Omega \setminus \Omega_\delta \text{ and } |z - z'| < \delta.$$

We have to prove that (3.2) also holds for $z, z' \in \Omega_\delta$ with $\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}$ replaced by $\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}$ in the right side. We use the notation

$$(3.3) \quad \begin{cases} u_{\frac{\delta}{2}} := \sup_{|\zeta| < \frac{\delta}{2}} u(z + \zeta), & z \in \Omega_\delta, \\ \tilde{u}_{\frac{\delta}{2}} := \frac{1}{\sigma_{2n-1} \left(\frac{\delta}{2}\right)^{2n-1}} \int_{b\mathbb{B}(z, \frac{\delta}{2})} u(\zeta) dS(\zeta), & z \in \Omega_\delta, \end{cases}$$

where $\sigma_{2n-1} \left(\frac{\delta}{2}\right)^{2n-1} = \text{Vol}(b\mathbb{B}(z, \frac{\delta}{2}))$. It is a classical consequence of Riesz Theorem that for a general plurisubharmonic function u , not necessarily C^2 , there is well defined Δu in the space of positive Borel measures. We use the notation $\|\Delta u\|^\Omega$ for the total mass of Δu on Ω .

Theorem 3.3. *We have*

$$(3.4) \quad \|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_\delta)} \lesssim \delta^{1-\epsilon} \|(-r)^{1+\epsilon} \Delta u\|^\Omega.$$

Proof. The proof is inspired to [7] Lemma 4.3; the novelty here consists in replacing δ^2 by $\delta^{1-\epsilon}(-r)^{1+\epsilon}$. We start from

$$(3.5) \quad \begin{aligned} \tilde{u}_{\frac{\delta}{2}}(z) - u(z) &\sim \frac{1}{\delta^{2n-1}} \int_{b\mathbb{B}(0, \frac{\delta}{2})} (u(z + \xi) - u(z)) dS(\xi) \\ &\sim \frac{1}{\delta^{2n-2}} \int_{b\mathbb{B}(0, \frac{\delta}{2})} dS(\xi) \int_0^1 \nabla u(z + s\xi) \cdot \frac{\xi}{\delta} ds \\ &\stackrel{\text{divergence thm.}}{=} \frac{1}{\delta^{2n-2}} \int_0^1 s ds \int_{\mathbb{B}(0, \frac{\delta}{2})} \Delta u(z + s\xi) \\ &\stackrel{s\xi=\zeta, s\delta=t}{\sim} \frac{1}{\delta^{2n-2}} \int_0^{\frac{\delta}{2}} \frac{t}{\delta^2} \frac{t^{-2n}}{\delta^{-2n}} dt \int_{\mathbb{B}(z, t)} \Delta u(\zeta). \end{aligned}$$

We denote by τ_ζ the translation by ζ and observe that $\tau_\zeta\Omega \subset \Omega_{\frac{\delta}{2}} \subset \subset \Omega$ for any $\zeta \in \mathbb{B}(0, t)$. Observing that the positive measure Δu has finite mass on compact subsets of Ω , in particular on $\Omega_{\frac{\delta}{2}}$, we get, for $t < \frac{\delta}{2}$

$$(3.6) \quad \int_{\Omega_\delta} dV(z) \int_{\mathbb{B}(z, t)} \Delta u \lesssim t^{2n} \int_{\Omega_{\frac{\delta}{2}}} \Delta u$$

We now perform integration $\int_{\Omega_\delta} \cdot dV(z)$ in both sides of (3.5), apply (3.6) and end up with

$$(3.7) \quad \begin{aligned} \int_{\Omega_\delta} (\tilde{u}_{\frac{\delta}{2}} - u)(z) dV(z) &\leq \int_0^{\frac{\delta}{2}} t^{-2n+1} t^{2n} dt \int_{\Omega_{\frac{\delta}{2}}} \Delta u \\ &\leq \int_0^{\frac{\delta}{2}} t \delta^{-(1+\epsilon)} dt \int_{\Omega_{\frac{\delta}{2}}} (-r)^{1+\epsilon} \Delta u \\ &\sim \delta^{1-\epsilon} \|(-r)^{1+\epsilon} \Delta u\|^{\Omega_{\frac{\delta}{2}}}. \end{aligned}$$

□

At this point, the problem is to prove the boundedness of $\|(-r)^{1+\epsilon} \Delta u\|^{\Omega_{\frac{\delta}{2}}}$ uniformly in δ . This holds (cf. Theorem 3.4 below) because of the presence of the factor $(-r)^{1+\epsilon}$. In absence of this factor, one should suppose from the beginning that Δu has finite total mass on Ω ; in turn, this would be a consequence of the hypothesis $\varphi \in C^{1,1}$ (cf. [7]).

Theorem 3.4. *We have*

$$(3.8) \quad \|(-r)^{1+\epsilon} \Delta u\|^\Omega \lesssim \|(-r)^{-1+\epsilon} u\|_{L^1(\Omega)}.$$

Proof. We take a system of smooth cut-offs $\chi_\nu(|z|) \in C_c^\infty(\mathbb{B}^{2n}(0, \frac{1}{\nu}))$, $\|\chi_\nu\|_{L^1} \equiv 1$, $\frac{1}{\nu} \rightarrow 0$, and regularize

$$u_\nu := \int_{\Omega} u(\tau) \chi_\nu(|z - \tau|) dV(\tau).$$

The u_ν 's belong to $C^\infty(\Omega)$, converge to u on Ω , and satisfy

$$(3.9) \quad \begin{cases} \sup_{\Omega_{\frac{1}{\nu}}} |\nabla u_\nu| = \sup_{\Omega_{\frac{1}{\nu}}} |\nabla(u * \chi_\nu)| \leq \nu \|u\|_{L^1(\Omega)} \\ \sup_{\Omega_{\frac{1}{\nu}}} u_\nu \leq c \quad \text{independent of } \nu. \end{cases}$$

Now that the u_ν 's are regular, the Δu_ν 's are well defined functions and hence we use the notation $\Delta u_\nu dV$ for the associated measures. We have

$$\begin{aligned}
(3.10) \quad & \int_{\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \Delta u_\nu dV(z) = \int_{\Omega_{\frac{1}{\nu}}} \operatorname{div}((-r)^{1+\epsilon} \nabla u_\nu) dV(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla u_\nu dV(z) \\
& \stackrel{\text{Stoke's}}{=} \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla u_\nu dV(z) \\
& = \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} \operatorname{div}((-r)^\epsilon (\nabla r u_\nu)) dV(z) \\
& \quad + \epsilon(1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} \nabla r \cdot \nabla r u_\nu dV(z) - (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \Delta r u_\nu dV(z) \\
& \stackrel{\text{Stoke's}}{=} \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{b\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla r u_\nu dV(z) \\
& \quad + \epsilon(1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} \nabla r \cdot \nabla r u_\nu dV(z) - (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \Delta r u_\nu dV(z) \\
& \stackrel{(3.9)}{\leq} O(\nu^{-\epsilon}) + (1+\epsilon)O(\nu^{-\epsilon}) + \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} |u_\nu| dV(z) + \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon |u_\nu| dV(z) \\
& \stackrel{\sim}{\leq} O(\nu^{-\epsilon}) + \|(-r)^{-1+\epsilon} u\|_{L^1(\Omega)}.
\end{aligned}$$

On the other hand, since u is plurisubharmonic, then Δu is a measure on Ω and $\Delta u_\nu dV \xrightarrow{\text{weakly}} \Delta u$. The conclusion follows from the following elementary Lemma

Lemma 3.5. *Assume $\Delta u_\nu \geq 0$ and*

$$\begin{cases} \int_{\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \Delta u_\nu dV \text{ are bounded} \\ \Delta u_\nu dV \xrightarrow{\text{weakly}} \Delta u. \end{cases}$$

Then

$$\int_{\Omega} (-r)^{1+\epsilon} \Delta u \text{ is bounded.}$$

The proof is just a consequence of the dominated convergence theorem for the sequence $(-r)^{1+\epsilon} \psi_\nu \Delta u_\nu dV \rightarrow (-r)^{1+\epsilon} \Delta u$ where ψ_ν are the characteristic functions of the sets $\Omega_{\frac{1}{\nu}}$. With Lemma 3.5 in our hands, we get the conclusion of the proof of Theorem 3.4.

□

End of Proof of Theorem 1.2. Again, we follow the guidelines of [7]. Along with \tilde{u}_δ defined by (3.3), we introduce $\hat{u}_\delta := \frac{1}{\sigma_{2n}\left(\frac{\delta}{2}\right)^{2n}} \int_{\mathbb{B}(z, \frac{\delta}{2})} u(\zeta) dV(\zeta)$, $z \in \Omega_\delta$. We recall that Lemma 4.2 of [7] states the equivalence between

$$(3.11) \quad \sup_{\Omega_\delta} u_{\frac{\delta}{2}} - u \lesssim \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}$$

and

$$(3.12) \quad \sup_{\Omega_\delta} \hat{u}_{\frac{\delta}{2}} - u \lesssim \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}.$$

On the other hand, on account of the obvious inequalities

$$\hat{u}_\delta \leq \tilde{u}_\delta \leq u_\delta,$$

we see that whatever of (3.11) and (3.12) is equivalent to

$$(3.13) \quad \sup_{\Omega_\delta} \tilde{u}_{\frac{\delta}{2}} - u \lesssim \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}.$$

We have thus to prove (3.13). To see it, we remark that

$$(3.14) \quad \begin{aligned} \|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_\delta)} &\lesssim \delta^{1-\epsilon} \|(-r)^{1+\epsilon} \Delta u\|_{\Omega_{\frac{\delta}{2}}}^{\frac{\delta}{2}} \\ &\stackrel{\text{Theorem 3.3}}{\lesssim} \delta^{1-\epsilon}. \end{aligned}$$

Theorem 3.4

By (3.2), we have for a suitable c

$$\tilde{u}_{\frac{\delta}{2}} \leq u_{\frac{\delta}{2}} \leq u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})} \quad \text{in a neighborhood of } b\Omega_\delta.$$

We can then define a plurisubharmonic function by

$$v_{\frac{\delta}{2}} := \begin{cases} \max(\tilde{u}_{\frac{\delta}{2}}, u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}) & \text{in } \Omega_\delta, \\ u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})} & \text{in } \Omega \setminus \Omega_\delta. \end{cases}$$

We apply the “stability estimate” to the pair $v_{\frac{\delta}{2}}$ and $u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}$ (since they coincide at $b\Omega$) and get

$$(3.15) \quad \begin{aligned} \sup_{\Omega_\delta} (v_{\frac{\delta}{2}} - (u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})})) &\stackrel{\text{stability estimate}}{\leq} \|v_{\frac{\delta}{2}} - (u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})})\|_{L^1(\Omega)}^\gamma \\ &\lesssim \|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_\delta)}^\gamma + (\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})})^\gamma \\ &\stackrel{(3.14)}{\lesssim} \delta^{(1-\epsilon)\gamma} + \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma} \sim \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma}. \end{aligned}$$

Since $\tilde{u}_{\frac{\delta}{2}} \leq v_{\frac{\delta}{2}}$ on Ω_{δ} , then

$$\tilde{u}_{\frac{\delta}{2}} - (u + c\delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})}) \underset{\sim}{\leq} \delta^{\min(\frac{2\gamma}{m}, \frac{\alpha}{m})\gamma},$$

and (3.13) follows.

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